Lecture 10

7.3 - Trigonometric Substitution

We want to compute integrals involving the expressions:

① $\sqrt{a^2-x^2}$ ② $\sqrt{x^2+a^2}$ ③ $\sqrt{x^2-a^2}$

(where $a>0$).

The philosophy is to replace what's under the integral with something squared: y^2. Then $\sqrt{y^2}=|y|$. The trig identities

$$\sin^2\theta+\cos^2\theta=1 \quad \& \quad 1+\tan^2\theta=\sec^2\theta$$

fit this bill perfectly.

① Integrals involving $\sqrt{a^2-x^2}$

Since a is a constant, the identity $\sin^2\theta+\cos^2\theta=1$ works here: if we let $x=a\sin\theta$, then we get

$$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2\theta}=\sqrt{a^2\cos^2\theta}=a|\cos\theta|$$

taking $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$, we can remove the absolute value bars on $\cos\theta$.

Ex: Compute $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

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② Integrals involving $\sqrt{x^2+a^2}$

The identity $1+\tan^2\theta=\sec^2\theta$ is well suited to this: letting $x=a\tan\theta$, then

$$\sqrt{x^2+a^2}=\sqrt{a^2\tan^2\theta+a^2}=\sqrt{a^2\sec^2\theta}=a|\sec\theta|$$

Restricting θ to: $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, we can again drop the absolute value bars.

Ex: Compute $\int \frac{dx}{\sqrt{4+3x^2}}$

③ Integrals involving $\sqrt{x^2-a^2}$

The identity $1+\tan^2\theta = \sec^2\theta$ is again useful here (write it $\sec^2\theta - 1 = \tan^2\theta$). Let $x = a\sec\theta$, then:

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} = a|\tan\theta|$$

To remove the absolute value bars, there are two good choices (the one you make is context dependent):

$$0 \leq \theta < \frac{\pi}{2} \quad \text{or} \quad \pi \leq \theta < \frac{3\pi}{2}$$

<u>Ex</u>: Compute $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-25}}$

Let's summarize this in a table:

Expression	Substitution	θ-range	Trig Identity
$\sqrt{a^2-x^2}$			
$\sqrt{a^2+x^2}$			
$\sqrt{x^2-a^2}$			

Sometimes, we can convert an integral into a trig sub integral by completing the square:

Ex: Compute $\int \sqrt{x^2+2x}\,dx$

Further Examples

Ex: $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}}$

Ex: $\displaystyle\int_0^3 \frac{t}{\sqrt{36-t^2}}\,dt$

Ex: $\int \frac{dv}{\sqrt{v^2-6v+13}}$

Ex: $\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}}$